Lecture 22: Determinants

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21.1 Definition

Let A be a nxn matrix.

There are 3 cases with respect to the determinant:

$n = 1$ $\quad A = [a]$ $\quad |A| = \det(A) = a$

$n = 2$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

$n \geq 3$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & . & . & \vdots \\ \vdots & \vdots & . & . & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$

Let A_{ij} be the $(n-1)x(n-1)$ matrix obtained from A by removing the i-th row and the j-th column.

<u>Alternative 1:</u>
Pick the i-th row of A, whatever i you like best.

$$\det(A) = (-1)^{i+1} \cdot a_{i1} \cdot \det(A_{i1}) + (-1)^{i+2} \cdot a_{i2} \cdot \det(A_{i2}) + \cdots + (-1)^{i+n} \cdot \mathrm{a_{in}} \cdot \det(\mathrm{A_{in}})$$

<u>Alternative 2:</u>
Pick the j-th column of A, whatever j you like best.

$$\det(A) = (-1)^{1+j} \cdot a_{1j} \cdot \det(A_{1j}) + (-1)^{2+j} \cdot a_{2j} \cdot \det(A_{2j}) + \cdots + (-1)^{n+j} \cdot \mathrm{a_{nj}} \cdot \det(\mathrm{A_{nj}})$$

<u>Remark</u>: det(A) does no depend on the alternative and the value chosen for i or j.

<u>Example</u>

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

expansion along first row

$$\det(A) = 2 \cdot \underbrace{\det \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{1} - 3 \cdot \underbrace{\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{-1} + 4 \cdot \underbrace{\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{0} - 5 \cdot \underbrace{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{1} = 0$$

Now, we have to deal with the four 3x3 matrices:

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \cancel{0 \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} - \cancel{0 \cdot \det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} + \underbrace{1 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{1}$$

$$= 1$$

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \cdot \cdots + 0 \cdot \cdots + 0 \cdot \cdots = 0$$

expansion along 3rd row

<u>note</u>: When computing the determinant, choose to expand along the row with the most number of zeroes.

How to cleverly compute the determinant of A:

$$\det(A) = -1 \cdot \det \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

4th

$$\det(A) = -1 \cdot \det \begin{bmatrix} 2^{+} & 3^{-} & 5^{+} \\ 1^{-} & 0^{+} & 1^{-} \\ 0^{+} & 1^{-} & 1^{+} \end{bmatrix}$$

4th row

3rd row

$$= -1 \cdot \left(-1 \cdot \underbrace{\det \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}}_{-3} + 1 \cdot \underbrace{\det \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}}_{-3}\right) = 0$$

21.2 Properties

Let A be an nxn matrix:

i) If A has a row or column of zeroes, then $\det(A) = 0$

ii) $\det(A^T) = \det(A)$

iii) If you have a triangular matrix (either all entries above or below the diagonal are zero), then $\det(A)$ is the product of the diagonal entries.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = 1 \cdot 4 \cdot 6 = 24$$

21.3 Determinants and Elementary Row Operations

Let A be an nxn matrix and let B be a matrix obtained from A by 1 ERO.
Then,

interchange 2 rows	$\det(B) = -\det(A)$
multiply one row by a scalar	$\det(B) = r \cdot \det(A)$
add a multiple of one row to another	$\det(B) = \det(\mathrm{A})$

This remains true if we replace "rows" with "columns"

Example

$$\det \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 5 \\ 4 & 0 & 3 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & -17 \end{bmatrix} = 34$$